Models of Set Theory II - Winter 2013

Prof. Dr. Peter Koepke, Dr. Philipp Schlicht Problem sheet 12

Problem 43 (4 Points)**.** If $T \subseteq {}^{<\omega}2$ is a tree, the n^{th} splitting level $\text{split}_n(T)$ of T is defined as the set of nodes $s \in T$ such that

(1) s is a *splitting node*, i.e. s has two incompatible direct successors.
(2) s is minimal such that there is no $t \in \bigcup_{i<n} \text{split}_i(T)$ with $s \leq t$.

Then $\text{split}(T) := \bigcup_{n\in\omega} \text{split}_n(T)$ is the set of splitting nodes in T.

A tree $T \subseteq {}^{<\omega}2$ is called *perfect* if $\text{split}(T)$ is cofinal in T. *Sacks forcing* $\mathbb{S}$ is the forcing whose conditions are perfect trees $T \subseteq {}^{<\omega}2$, ordered by $S \leq T :\Leftrightarrow S \subseteq T$. Let $S \leq_n T$ for $S, T \in \mathbb{S}$ if $S \leq T$ and $\text{split}_i(S) = \text{split}_i(T)$ for all $i < n$.

Show that Sacks forcing with $(\leq_n)_{n\in\omega}$ satisfies the fusion condition for Axiom A in Problem 41 (i).

Problem 44 (8 Points)**.** Suppose that P is a forcing and $p \in P$. Consider the following game $G'(P,p)$ for two players with ω moves. In round n, player I plays a maximal antichain A_n in P and then player II responds by playing countable sets $B_0^n \subseteq A_0$, ... $B_n^n \subseteq A_n$. Player II wins if there is a condition $q \leq p$ such that for all $n \in \omega$, the set $\bigcup_{n\leq m\in\omega} B_n^m$ is predense below q.

Show that Player II has a winning stategy for $G'(P,p)$ if and only if for all cardinals λ, player II has a winning strategy for the game $G_\lambda(P,p)$ in Problem 46.

(*Hint: Suppose that player II has a winning strategy in $G'(P,p)$. We consider a run of $G_\lambda(P,p)$. When player I plays $\dot\alpha_n$ in $G_\lambda(P,p)$, choose a maximal antichain A_n such that every $p \in A_n$ decides $\dot\alpha_n$. Let player I play A_n in $G'(P,p)$ and as the next move for II in $G_\lambda(P,p)$, construct a set of ordinals from the answer of II in $G'(P,p)$.*

For the other implication, suppose that $(P, <_P, 1_P) = (\mu, <_P, 1_P)$ and let $\lambda = \max\{\mu^+, \omega\}$. Suppose that player II has a winning strategy in $G_\lambda(P,p)$. We consider a run of $G'(P,p)$. When player I plays A_n in $G'(P,p)$, let player I play $\dot\alpha_n := \{(\check\beta, \alpha) \mid \alpha \in A_n,\ \beta < \mu \cdot n + \alpha\}$. If the answer of II in $G_\lambda(P,p)$ is C_n, let II play $B_k^n = \{\alpha < \mu \mid \alpha \in A_k,\ \mu \cdot k + \alpha \in C_n\}$ for $k \leq n$ in $G'(P,p)$.

Show that these instructions define winning strategies for II in $G_\lambda(P,p)$ and $G'(P,p)$, respectively.)

Problem 45 (6 Points)**.** Suppose that P is a forcing and player II has a winning strategy in $G'(P,p)$ for all $p \in P$. Show that P is proper.

(*Hint: Suppose that σ is a winning strategy for player II, and let λ be sufficiently large with $\sigma \in H_\lambda$. Show that for every $M \prec H_\lambda$ with $P, p, \sigma \in M$ there is an (M,P)-generic condition $q \leq p$, by defining a run of $G'(P,p)$ in which I plays all maximal antichains $A \in M$.*)

Please hand in your solutions on Wednesday, January 29 before the lecture.